

Improved stability estimates on general scalar balance laws

Magali Lécureux-Mercier^a

October 26, 2010

Abstract

Consider the general scalar balance law $\partial_t u + \text{Div} f(t, x, u) = F(t, x, u)$ in several space dimensions. The aim of this note is to improve the results of Colombo, Mercier, Rosini who gave an estimate of the dependence of the solutions from the flow f and from the source F . The improvements are twofold: first the expression of the coefficients in these estimates are more precise; second, we eliminate some regularity hypotheses thus extending significantly the applicability of our estimates.

2000 Mathematics Subject Classification: 35L65.

Keywords: Multi-dimensional scalar conservation laws, Kružkov entropy solutions, **BV** estimate.

1 Introduction

We consider here the Cauchy problem for the general scalar balance law

$$\begin{cases} \partial_t u + \text{Div} f(t, x, u) = F(t, x, u) & (t, x) \in \mathbb{R}_+^* \times \mathbb{R}^N \\ u(0, x) = u_0(x) & x \in \mathbb{R}^N. \end{cases} \quad (1.1)$$

This kind of equation has already been intensively studied: a fundamental result is the one of S. N. Kružkov [10, Theorem 1 & 5], stating the existence and uniqueness of a weak entropy solution for an initial data $u_0 \in \mathbf{L}^\infty(\mathbb{R}^N, \mathbb{R})$. Furthermore, Kružkov describes the dependence of the solutions with respect to the initial condition: if u_0 and v_0 are two initial data, then the associated entropy solutions u and v satisfy

$$\|(u - v)(t)\|_{\mathbf{L}^1} \leq e^{\gamma t} \|u_0 - v_0\|_{\mathbf{L}^1}, \quad \text{with } \gamma = \|\partial_u F\|_{\mathbf{L}^\infty}. \quad (1.2)$$

Furthermore, a huge literature on this subject is available in the special case the flow f depend only on u and not on the variables t and x and there is no source $F = 0$ (see for example [3, 8, 12, 13]).

We are interested here in the dependence of the solution with respect to flow f and source F in the case these functions depend on the three variables t , x and u .

This dependence with respect to flow and source has already been investigated: this question was first addressed from the point of view of numerical analysis by B. Lucier [11] who studied the case of an homogeneous flow ($f(u)$), without source term ($F = 0$).

^aLaboratoire MAPMO, Université d'Orléans, UFR Sciences, Bâtiment de mathématiques - Rue de Chartres, B.P. 6759 - 45067 Orléans cedex 2 France

More recently F. Bouchut & B. Perthame [2] improved this result, always in the case of an homogeneous flow and without source. G.-Q. Chen & K. Karlsen [4] also studied this dependence, for a flow depending also on x , but the estimate they obtained was depending on an a priori (unknown) bound on $\text{TV}(u(t))$.

The purpose of the present paper is to improve the recent result of R. Colombo, M. Mercier & M. Rosini [7], which provided an estimate of the total variation in the general case (with flow and source depending on the three variables t, x and u) and of the \mathbf{L}^1 distance between solutions. In particular, this estimate can be compared to the one of Kruřkov (1.2) that give a bound on the \mathbf{L}^1 distance between solutions with different initial data (but with same flow and source). The estimates (1.2) and [7, Theorem 2.6] look similar but in [7], the coefficient γ given by Kruřkov in (1.2) is replaced by $\kappa = 2N\|\nabla\partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty}$. Consequently, we do not recover (1.2) from [7] in the case $F = 0$ (because $\gamma = 0$ whereas $\kappa = 2N\|\nabla\partial_u f\|_{\mathbf{L}^\infty} \neq 0$ a priori).

In the same setting as in [7, 10], we provide here an estimate on the total variation of the solution to (1.1), and on the dependence of the solutions to (1.1) on the flow f , on the source F , with better hypotheses and coefficients than in [7]. The advances are twofold. Firstly, we relax hypotheses, and thus widely extend the usability of our results. More precisely, we require here less regularity in time than in [7], which is very useful for applications (see [5, 6]). Furthermore, we recover the same estimate as Kruřkov when we consider the dependence toward initial conditions only.

This note is organized as follows. In Section 2 we state the main results and compare them to those in [7]. In Section 3, we give some tools on functions with bounded variations; in Sections 4 and 5 we prove Theorems 2.2 and 2.4; finally Section 6 contains some technical lemmas used in the preceding sections.

2 Main results

We shall use the notations $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}_+^* = (0, +\infty)$. Below, N is a positive integer, $\Omega = \mathbb{R}_+^* \times \mathbb{R}^N \times \mathbb{R}$; for any positive T, U we denote $\Omega_T^U = [0, T] \times \mathbb{R}^N \times [-U, U]$; $B(x, r)$ stands for the ball in \mathbb{R}^N with center $x \in \mathbb{R}^N$ and radius $r > 0$ and $\text{Supp}(u)$ stands for the support of u . The volume of the unit ball $B(0, 1)$ is ω_N . For notational simplicity, we set $\omega_0 = 1$. The following induction formula gives ω_N in terms of the Wallis integral W_N :

$$\frac{\omega_N}{\omega_{N-1}} = 2W_N \quad \text{where} \quad W_N = \int_0^{\pi/2} (\cos \theta)^N d\theta. \quad (2.1)$$

In the present work, $\mathbf{1}_A$ is the characteristic function of the set A , and δ_t is the Dirac measure centered at t . Besides, for a vector valued function $f = f(x, u)$ with $u = u(x)$, $\text{Div} f$ stands for the total divergence. On the other hand, $\text{div} f$, respectively ∇f , denotes the partial divergence, respectively gradient, with respect to the space variables. Moreover, ∂_u and ∂_t are the usual partial derivatives. Thus, $\text{Div} f = \text{div} f + \partial_u f \cdot \nabla u$.

The following sets of assumptions on f and F will be of use below.

$$\begin{aligned}
(\mathbf{H1}^*) \quad & \begin{cases} f \in \mathcal{C}^0(\Omega; \mathbb{R}^N), & F \in \mathcal{C}^0(\Omega; \mathbb{R}), \\ f, F \text{ have continuous derivatives } \partial_u f, \partial_u \nabla f, \nabla^2 f, \partial_u F, \nabla F; \\ \text{for all } U, T > 0, & \partial_u f \in \mathbf{L}^\infty(\Omega_T^U; \mathbb{R}^N), \quad F - \operatorname{div} f \in \mathbf{L}^\infty(\Omega_T^U; \mathbb{R}). \end{cases} \\
(\mathbf{H2}^*) \quad & \begin{cases} \text{for all } U, T > 0, & \nabla \partial_u f \in \mathbf{L}^\infty(\Omega_T^U; \mathbb{R}^{N \times N}), \quad \partial_u F \in \mathbf{L}^\infty(\Omega_T^U; \mathbb{R}), \\ \int_0^T \int_{\mathbb{R}^N} \|\nabla(F - \operatorname{div} f)(t, x, \cdot)\|_{\mathbf{L}^\infty([-U, U]; \mathbb{R}^N)} dx dt < \infty. \end{cases} \\
(\mathbf{H3}^*) \quad & \begin{cases} \text{for all } U, T > 0 & \partial_u f \in \mathbf{L}^\infty(\Omega_T^U; \mathbb{R}^N), \quad \partial_u F \in \mathbf{L}^\infty(\Omega_T^U; \mathbb{R}), \\ \int_0^T \int_{\mathbb{R}^N} \|(F - \operatorname{div} f)(t, x, \cdot)\|_{\mathbf{L}^\infty([-U, U]; \mathbb{R})} dx dt < +\infty. \end{cases}
\end{aligned}$$

Comparing these sets of hypotheses to **(H1)**, **(H2)** and **(H3)** in [7], we note that

- no derivatives in time are now needed;
- the \mathbf{L}^∞ norm are now taken on the domain $\Omega_T^U = [0, T] \times \mathbb{R}^N \times [-U, U]$ that is smaller than $\Omega = \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}$, that was the domain considered in [7].

Let us recall the fundamental theorem

Theorem 2.1 (Kruřkov [10]). *Assume **(H1)*** hold. Then, for any $u_0 \in \mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R})$, there exists a unique weak entropy solution u to (1.1) in $\mathbf{L}^\infty(\mathbb{R}_+; \mathbf{L}_{loc}^1(\mathbb{R}^N; \mathbb{R}))$ continuous from the right. Moreover, if a sequence $u_0^n \in \mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R})$ converges to u_0 in \mathbf{L}_{loc}^1 , then for all $t > 0$ the corresponding solutions $u^n(t)$ converge to $u(t)$ in \mathbf{L}_{loc}^1 .*

2.1 Estimate on the Total Variation

We give here a result similar to the one obtained by Colombo, Mercier and Rosini [7, Theorem 2.5], but under weaker assumptions.

Theorem 2.2. *Assume that **(H1)*** and **(H2)*** hold. Let $u_0 \in (\mathbf{L}^\infty \cap \mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^N; \mathbb{R})$. Then, the weak entropy solution u of (1.1) satisfies $u(t) \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$ for all $t > 0$. Let T_0 be real positive. Let us denote $\mathcal{U} = \|u\|_{\mathbf{L}^\infty([0, T_0] \times \mathbb{R}^N)}$, $U_t = \sup_{y \in \mathbb{R}^N} |u(t, y)|$, $\mathcal{S}_{T_0}(u) = \bigcup_{t \in [0, T_0]} \operatorname{Supp}(u(t))$ and*

$$\Sigma_{T_0}^u = [0, T_0] \times \mathcal{S}_{T_0}(u) \times [-\mathcal{U}, \mathcal{U}], \quad (2.2)$$

$$\kappa_0^* = (2N + 1) \|\nabla \partial_u f\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u; \mathbb{R}^{N \times N})} + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u; \mathbb{R})} \quad (2.3)$$

then for all $T \in [0, T_0]$, with W_N as in (2.1),

$$\operatorname{TV}(u(T)) \leq \operatorname{TV}(u_0) e^{\kappa_0^* T} + N W_N \int_0^T e^{\kappa_0^*(T-t)} \int_{\mathbb{R}^N} \|\nabla(F - \operatorname{div} f)(t, x, \cdot)\|_{\mathbf{L}^\infty([-U_t, U_t]; \mathbb{R})} dx dt. \quad (2.4)$$

Remark 2.3. Note that, with $c = \|\partial_u f\|_{\mathbf{L}^\infty(\Omega_{T_0}^U)}$, we have $\operatorname{Supp} u(t) \subset \operatorname{Supp} u_0 + B(0, ct)$. Consequently,

$$\mathcal{S}_{T_0}(u) \subset \operatorname{Supp} u_0 + B(0, c T_0).$$

We can note here several improvements with respect to [7, Theorem 2.5]. First, as we already noted, the set of hypotheses is weaker since we do not require f to be \mathcal{C}^2 and F to be \mathcal{C}^1 with respect to the time variable: they only have to be continuous in time, which is useful in applications, see for example [5].

A second improvement stands in the \mathbf{L}^∞ norms, that are taken on smaller domains than in [7].

Last, the expression of the coefficient κ_0^* that does not content any longer the constant NW_N . Indeed, in [7, Theorem 2.5] it was given by

$$\kappa_0 = NW_N \left((2N+1) \|\nabla \partial_u f\|_{\mathbf{L}^\infty(\Omega; \mathbb{R}^{N \times N})} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \right)$$

Besides, it does not seem possible to erase the coefficient NW_N completely from the expression (2.4), except in the case F and f do not depend on u , see Remark 4.1.

2.2 Stability of Solutions with Respect to Flow and Source

We want now to estimate the difference $u - v$, where

- u is the solution of (1.1) with flow f , source F and initial condition u_0 ,
- v is the solution of (1.1) with flow g , source G and initial condition v_0 .

We search for an estimate of $u - v$ in term of $f - g$, $F - G$ and $u_0 - v_0$.

F. Bouchut & B. Perthame in [2] obtained such an estimate in the particular case f, g depend only on u and $F = G = 0$. The following result is an improvement of the result of R. Colombo, M. Mercier and M. Rosini [7, Theorem 2.6], in which we gave a similar result under stronger assumptions and with a coefficient κ^* that was not compatible with the result of Kruřkov (1.2).

Theorem 2.4. *Let $(f, F), (g, G)$ satisfy $(\mathbf{H1}^*)$, (f, F) satisfy $(\mathbf{H2}^*)$ and $(f - g, F - G)$ satisfy $(\mathbf{H3}^*)$. Let $u_0, v_0 \in \mathbf{L}^\infty \cap \mathbf{L}^1 \cap \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$. Let $T > 0$, we denote $\mathcal{V} = \max(\|u\|_{\mathbf{L}^\infty([0, T] \times \mathbb{R}^N)}, \|v\|_{\mathbf{L}^\infty([0, T] \times \mathbb{R}^N)})$, $V_t = \sup_{y \in \mathbb{R}^N} (|u(t, y)|, |v(t, y)|)$. Let also introduce $\mathcal{S}_T(u, v) = \bigcup_{t \in [0, T]} (\text{Supp } u(t) \cup \text{Supp } v(t))$ and*

$$\Sigma_T^{u, v} = [0, T] \times \mathcal{S}_T(u, v) \times [-\mathcal{V}, \mathcal{V}].$$

Furthermore, we define and κ_0^* , U_t , Σ_T^u as in (2.3) and

$$\kappa^* = \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma_T^{u, v}; \mathbb{R})}, \quad M = \|\partial_u g\|_{\mathbf{L}^\infty(\Omega_T^v; \mathbb{R}^N)}.$$

Then, for any $R > 0$ and $x_0 \in \mathbb{R}^N$, the following estimate holds:

$$\begin{aligned} & \int_{\|x-x_0\| \leq R} |u(T, x) - v(T, x)| dx \leq e^{\kappa^* T} \int_{\|x-x_0\| \leq R+MT} |u_0(x) - v_0(x)| dx \\ & + \frac{e^{\kappa_0^* T} - e^{\kappa^* T}}{\kappa_0^* - \kappa^*} \text{TV}(u_0) \|\partial_u(f - g)\|_{\mathbf{L}^\infty(\Sigma_T^u; \mathbb{R}^N)} \\ & + NW_N \left(\int_0^T \frac{e^{\kappa_0^*(T-t)} - e^{\kappa^*(T-t)}}{\kappa_0^* - \kappa^*} \int_{\mathbb{R}^N} \|\nabla(F - \text{div } f)(t, x, \cdot)\|_{\mathbf{L}^\infty([-U_t, U_t])} dx dt \right) \\ & \quad \times \|\partial_u(f - g)\|_{\mathbf{L}^\infty(\Sigma_T^u; \mathbb{R}^N)} \\ & + \int_0^T e^{\kappa^*(T-t)} \int_{\|x-x_0\| \leq R+M(T-t)} \left\| ((F - G) - \text{div}(f - g))(t, x, \cdot) \right\|_{\mathbf{L}^\infty([-V_t, V_t])} dx dt. \end{aligned}$$

Remark 2.5. Note as above that, with $c' = \max(\|\partial_u f\|_{\mathbf{L}^\infty(\Omega_{T_0}^u)}, \|\partial_u g\|_{\mathbf{L}^\infty(\Omega_{T_0}^v)})$, we have $\text{Supp } u(t) \subset \text{Supp } u_0 + B(0, c' t)$ and $\text{Supp } v(t) \subset \text{Supp } v_0 + B(0, c' t)$. Consequently,

$$\mathcal{S}_T(u, v) \subset (\text{Supp } u_0 \cup \text{Supp } v_0) + B(0, c' T).$$

As above, we can note some improvements with respect to [7, Theorem 2.6]:

- The hypotheses are weaker: no derivative in time is needed for f and F .
- The \mathbf{L}^∞ norms are taken on smaller domains.
- The coefficient κ^* is better than the κ given in [7, Theorem 2.6] by

$$\kappa = 2N \|\nabla \partial_u f\|_{\mathbf{L}^\infty(\Omega; \mathbb{R}^{N \times N})} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} + \|\partial_u(F - G)\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})}.$$

Indeed, κ^* only depends on F , which is consistent with the previous Kruřkov's result (1.2), whereas κ was also depending on f .

Note that, denoting $h = (2N + 1) \|\nabla \partial_u f\|_{\mathbf{L}^\infty(\Sigma_T^u; \mathbb{R}^{N \times N})}$, we have $\kappa^* + h = \kappa_0^*$ and

$$\frac{e^{\kappa_0^* t} - e^{\kappa^* t}}{\kappa_0^* - \kappa^*} = e^{\kappa^* t} \frac{e^{ht} - 1}{h} \leq t e^{(\kappa^* + h)t}.$$

In the case $\Sigma_T^{u,v} = \Sigma_T^u$, we conclude by $t e^{(\kappa^* + h)t} = t e^{\kappa_0^* t}$.

In the case, $\Sigma_T^u \neq \Sigma_T^{u,v}$, we can replace κ_0^* by

$$\kappa_1 = (2N + 1) \|\nabla \partial_u f\|_{\mathbf{L}^\infty(\Sigma_T^u; \mathbb{R}^{N \times N})} + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma_T^{u,v}; \mathbb{R})}.$$

Then we have $\kappa_1 = \kappa^* + h$ and $t e^{(\kappa^* + h)t} = t e^{\kappa_1 t}$.

In all cases, we obtain $\frac{e^{\kappa_0^* t} - e^{\kappa^* t}}{\kappa_0^* - \kappa^*} \leq t e^{\kappa_1 t}$, and the estimate of Theorem 2.4 becomes

$$\begin{aligned} & \int_{\|x-x_0\| \leq R} |u(T, x) - v(T, x)| dx \leq e^{\kappa^* T} \int_{\|x-x_0\| \leq R+MT} |u_0(x) - v_0(x)| dx \\ & + T e^{\kappa_1 T} \text{TV}(u_0) \|\partial_u(f - g)\|_{\mathbf{L}^\infty(\Sigma_T^u; \mathbb{R}^N)} \\ & + NW_N \left(\int_0^T (T-t) e^{\kappa_1 (T-t)} \int_{\mathbb{R}^N} \|\nabla(F - \text{div } f)(t, x, \cdot)\|_{\mathbf{L}^\infty([-U_t, U_t])} dx dt \right) \\ & \quad \times \|\partial_u(f - g)\|_{\mathbf{L}^\infty(\Sigma_T^u; \mathbb{R}^N)} \\ & + \int_0^T e^{\kappa^* (T-t)} \int_{\|x-x_0\| \leq R+M(T-t)} \left\| ((F - G) - \text{div}(f - g))(t, x, \cdot) \right\|_{\mathbf{L}^\infty([-V_t, V_t])} dx dt. \end{aligned}$$

3 Tools on functions with bounded variation

Recall the following theorem (see [1, Theorem 3.9 and Remark 3.10]):

Theorem 3.1. *Let $u \in \mathbf{L}_{loc}^1(\mathbb{R}^N; \mathbb{R})$. Then $u \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$ if and only if there exists a sequence (u_n) in $\mathcal{C}^\infty(\mathbb{R}^N; \mathbb{R})$ converging to u in \mathbf{L}_{loc}^1 and satisfying*

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} \|\nabla u_n(x)\| dx = L \quad \text{with} \quad L < \infty.$$

Moreover, $\text{TV}(u)$ is the smallest constant L for which there exists a sequence as above.

Let us also recall the following property of any function $u \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$:

$$\int_{\mathbb{R}^N} |u(x) - u(x - z)| \, dx \leq \|z\| \, \text{TV}(u) \quad \text{for all } z \in \mathbb{R}^N. \quad (3.1)$$

For a proof, see [1, Remark 3.25].

Now, in a similar way as J. Dávila [9], we prove the following proposition, which is an improvement of [7, Proposition 4.3]. Indeed, in [7, Proposition 4.3], the equality (3.3) is valid only for $u \in \mathcal{C}^1$. In the present proposition we extend this result to all $u \in \mathbf{BV}$.

Proposition 3.2. *Let $\rho_1 \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}_+)$ with $\text{Supp } \rho_1 \subset [-1, 1]$. Let $u \in \mathbf{L}_{loc}^1(\mathbb{R}^N; \mathbb{R})$. For all $\lambda > 0$, we introduce ρ_λ such that $\rho_\lambda(x) = \frac{1}{\lambda^N} \rho_1\left(\frac{\|x\|}{\lambda}\right)$. Assume that there exists a constant \tilde{C} such that for all λ, R positive,*

$$\frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{B(x_0, R)} |u(x) - u(x - z)| \rho_\lambda(z) \, dx \, dz \leq \tilde{C}. \quad (3.2)$$

Then $u \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$ and

$$\text{TV}(u) = \frac{1}{C_1} \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(x - z)| \rho_\lambda(z) \, dx \, dz, \quad (3.3)$$

where

$$C_1 = \int_{\mathbb{R}^N} |x_1| \rho_1(\|x\|) \, dx. \quad (3.4)$$

Proof.

Note that the first part of the proof is the same as the first part of the proof of [7, Proposition 4.3]. We introduce a regularisation of u : $u_h = u * \mu_h$, with $\mu_h(x) = \mu_1(\|x\|/h) / h^N$, where μ_1 is defined as in (6.1). We note that $u_h \in \mathcal{C}^\infty(\mathbb{R}^N; \mathbb{R})$ and that u_h tends to u in \mathbf{L}_{loc}^1 when $h \rightarrow 0$. Furthermore, for R and h positive, by change of variables we get

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{B(x_0, R-h)} \left| \int_0^1 \nabla u_h(x - \lambda s z) \cdot z \, ds \right| \rho_1(\|z\|) \, dx \, dz \\ &= \frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{B(x_0, R-h)} |u_h(x) - u_h(x - \lambda z)| \rho_1(\|z\|) \, dx \, dz \\ &\leq \frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{B(x_0, R)} |u(x) - u(x - \lambda z)| \rho_1(\|z\|) \, dx \, dz \\ &\leq \tilde{C}. \end{aligned}$$

Making $R \rightarrow \infty$ and using the Dominated Convergence Theorem when $\lambda \rightarrow 0$, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\nabla u_h(x) \cdot z| \rho_1(\|z\|) \, dx \, dz \\ &\leq \liminf_{\lambda \rightarrow 0} \frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{B(x_0, R)} |u(x) - u(x - \lambda z)| \rho_1(\|z\|) \, dx \, dz. \end{aligned}$$

Remark that for fixed $x \in \mathbb{R}^N$, when $\nabla u_h(x) \neq 0$, the scalar product $\nabla u_h(x) \cdot z$ is positive (respectively, negative) when z is in a half-space, say H_x^+ (respectively, H_x^-). We

can write $z = \alpha \frac{\nabla u_h(x)}{\|\nabla u_h(x)\|} + w$, with $\alpha \in \mathbb{R}$ and w in the hyperplane $H_x^\circ = \nabla u_h(x)^\perp$. Hence

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla u_h(x) \cdot z| \mu_1(\|z\|) dz &= \int_{H_x^+} \nabla u_h(x) \cdot z \mu_1(\|z\|) dz + \int_{H_x^-} \nabla u_h(x) \cdot (-z) \mu_1(\|z\|) dz \\
&= 2 \int_{H_x^+} \nabla u_h(x) \cdot z \mu_1(\|z\|) dz \\
&= 2 \int_{\mathbb{R}_+^*} \int_{H_x^\circ} \alpha \|\nabla u_h(x)\| \mu_1(\sqrt{\alpha^2 + \|w\|^2}) dw d\alpha \\
&= \int_{\mathbb{R}} \int_{H_x^\circ} |\alpha| \|\nabla u_h(x)\| \mu_1(\sqrt{\alpha^2 + \|w\|^2}) dw d\alpha \\
&= \|\nabla u_h(x)\| \int_{\mathbb{R}^N} |z_1| \mu_1(\|z\|) dz.
\end{aligned}$$

So we obtain

$$\mathrm{TV}(u) \leq \frac{1}{C_1} \liminf_{\lambda \rightarrow 0} \frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(x - z)| \rho_\lambda(z) dx dz \leq \frac{\tilde{C}}{C_1}. \quad (3.5)$$

Now, let (u_n) be a sequence of functions in $\mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R})$ converging to u in \mathbf{L}_{loc}^1 and such that $\int_{\mathbb{R}^N} \|\nabla u_n(x)\| dx$ converges to $\mathrm{TV}(u)$ when $n \rightarrow \infty$. Then, doing the same computation as above, we obtain

$$\begin{aligned}
&\frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{B(x_0, R)} |u_n(x) - u_n(x - \lambda z)| \rho_1(\|z\|) dx dz \\
&\leq \int_{\mathbb{R}^N} \int_{B(x_0, R)} \int_0^1 |\nabla u_n(x - \lambda s z) \cdot z| \rho_1(\|z\|) ds dx dz \\
&\leq \int_{\mathbb{R}^N} \int_0^1 \int_{B(x_0, R+\lambda)} |\nabla u_n(x') \cdot z| \rho_1(\|z\|) dx' ds dz \\
&= \int_{B(x_0, R+\lambda)} \|\nabla u_n(x)\| C_1 dx \\
&\leq C_1 \mathrm{TV}(u_n, B(x_0, R+\lambda)).
\end{aligned}$$

Taking $R \rightarrow \infty$ and then $n \rightarrow \infty$, we have consequently

$$\frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(x - \lambda z)| \rho_1(\|z\|) dx dz \leq C_1 \mathrm{TV}(u).$$

Then, we take the supremum limit when λ goes to 0. We obtain

$$\limsup_{\lambda \rightarrow 0} \frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(x - z)| \rho_\lambda(z) dx dz \leq C_1 \mathrm{TV}(u). \quad (3.6)$$

We conclude the proof by reassembling (3.5) and (3.6). \square

4 Proof of Theorem 2.2

The following proof is quite similar to the one of [7, Theorem 2.5]. The differences come from the use of Proposition 3.2 instead of [7, Proposition 4.3] and from avoiding the derivatives in time to appear. In order to be clear, we rewrite here most of the steps of the proof. In particular, the beginning of the proof is similar to [7, proof of Theorem 2.5] up to (4.13).

Proof of Theorem 2.2. First, we assume that $u_0 \in \mathcal{C}^1(\mathbb{R}^N; \mathbb{R})$. The general case will be considered only at the end of this proof.

By Kružkov Theorem [10, Theorem 5 & Section 5 Remark 4], the set of hypotheses **(H1*)** gives us existence and uniqueness of a weak entropy solution for any initial condition $u_0 \in \mathbf{L}^\infty \cap \mathbf{L}^1(\mathbb{R}^N; \mathbb{R})$. Let u be the weak entropy solution to (1.1) associated to $u_0 \in (\mathbf{L}^\infty \cap \mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^N; \mathbb{R})$. Let us denote $u = u(t, x)$ and $v = u(s, y)$ for $(t, x), (s, y) \in \mathbb{R}_+^* \times \mathbb{R}^N$. Then, for all $k, l \in \mathbb{R}$ and for all test functions $\varphi = \varphi(t, x, s, y)$ in $\mathcal{C}_c^1((\mathbb{R}_+^* \times \mathbb{R}^N)^2; \mathbb{R}_+)$, we have

$$\int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \left[(u - k) \partial_t \varphi + (f(t, x, u) - f(t, x, k)) \nabla_x \varphi + (F(t, x, u) - \operatorname{div} f(t, x, k)) \varphi \right] \times \operatorname{sign}(u - k) dx dt \geq 0 \quad (4.1)$$

for all $(s, y) \in \mathbb{R}_+^* \times \mathbb{R}^N$, and

$$\int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \left[(v - l) \partial_s \varphi + (f(s, y, v) - f(s, y, l)) \nabla_y \varphi + (F(s, y, v) - \operatorname{div} f(s, y, l)) \varphi \right] \times \operatorname{sign}(v - l) dy ds \geq 0 \quad (4.2)$$

for all $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}^N$. Let $\Phi \in \mathcal{C}_c^\infty(\mathbb{R}_+^* \times \mathbb{R}^N; \mathbb{R}_+)$, $\Psi \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{R}^N; \mathbb{R}_+)$ and set

$$\varphi(t, x, s, y) = \Phi(t, x) \Psi(t - s, x - y). \quad (4.3)$$

Observe that $\partial_t \varphi + \partial_s \varphi = \Psi \partial_t \Phi$, $\nabla_x \varphi = \Psi \nabla_x \Phi + \Phi \nabla_x \Psi$, $\nabla_y \varphi = -\Phi \nabla_x \Psi$. Choose $k = v(s, y)$ in (4.1) and integrate with respect to (s, y) . Analogously, take $l = u(t, x)$ in (4.2) and integrate with respect to (t, x) . Summing the obtained inequalities, we obtain

$$\begin{aligned} & \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \operatorname{sign}(u - v) \left[(u - v) \Psi \partial_t \Phi + (f(t, x, u) - f(t, x, v)) \cdot (\nabla \Phi) \Psi \right. \\ & \quad \left. + (f(s, y, v) - f(s, y, u) - f(t, x, v) + f(t, x, u)) \cdot (\nabla \Psi) \Phi \right. \\ & \quad \left. + (F(t, x, u) - F(s, y, v) + \operatorname{div} f(s, y, u) - \operatorname{div} f(t, x, v)) \varphi \right] dx dt dy ds \geq 0. \end{aligned} \quad (4.4)$$

Introduce a family of functions $\{Y_\vartheta\}_{\vartheta > 0}$ such that for any $\vartheta > 0$:

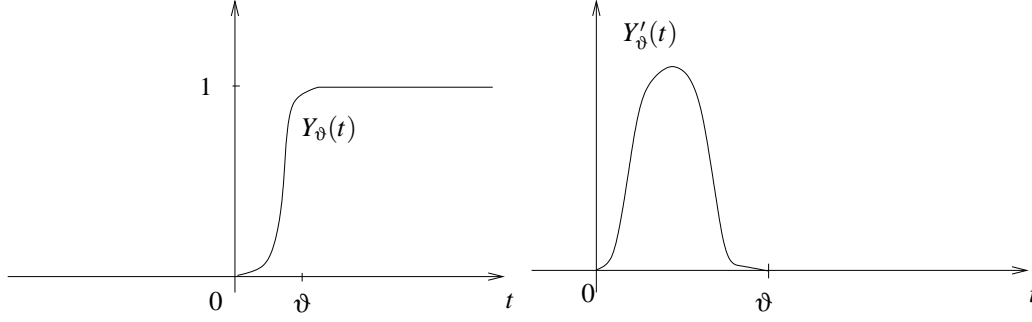
$$Y_\vartheta(t) = \int_{-\infty}^t Y'_\vartheta(s) ds, \quad Y'_\vartheta(t) = \frac{1}{\vartheta} Y' \left(\frac{t}{\vartheta} \right), \quad Y' \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{R}), \quad (4.5)$$

$$\operatorname{Supp}(Y') \subset]0, 1[, \quad Y' \geq 0, \quad \int_{\mathbb{R}} Y'(s) ds = 1.$$

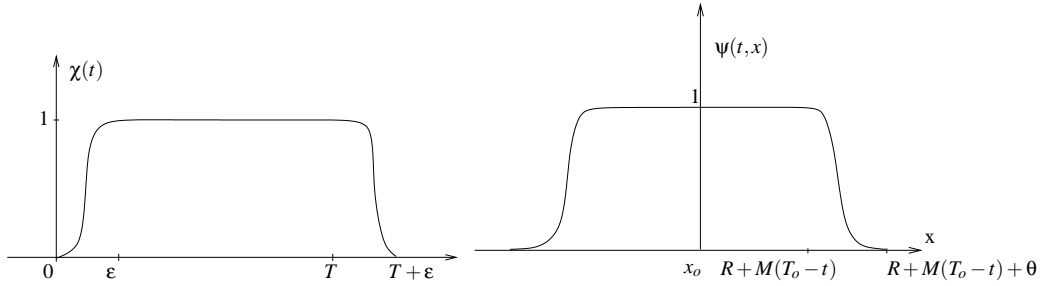
Let $T_0 > 0$, $\mathcal{U} = \|u\|_{\mathbf{L}^\infty([0, T_0] \times \mathbb{R}^N; \mathbb{R})}$ and $M = \|\partial_u f\|_{\mathbf{L}^\infty(\Omega_{T_0}^u; \mathbb{R}^N)}$ which is bounded by **(H1*)**.

Let us also define, for $\varepsilon, \theta, R > 0$, $x_0 \in \mathbb{R}^N$, (see Figure 2):

$$\chi(t) = Y_\varepsilon(t) - Y_\varepsilon(t - T) \quad \text{and} \quad \psi(t, x) = 1 - Y_\theta(\|x - x_0\| - R - M(T_0 - t)) \geq 0, \quad (4.6)$$

Figure 1: Graphs of Y_ϑ , left, and of Y'_ϑ , right.

where we also need the compatibility conditions $T_0 \geq T$ and $M\varepsilon \leq R + M(T_0 - T)$.

Figure 2: Graphs of χ , left, and of ψ , right.

Observe that $\chi \rightarrow \mathbf{1}_{[0,T]}$ and $\chi' \rightarrow \delta_0 - \delta_T$ as ε tends to 0. On χ and ψ we use the bounds

$$\chi \leq \mathbf{1}_{[0, T+\varepsilon]} \quad \text{and} \quad \mathbf{1}_{B(x_0, R+M(T_0-t))} \leq \psi \leq \mathbf{1}_{B(x_0, R+M(T_0-t)+\theta)}.$$

In (4.4), choose $\Phi(t, x) = \chi(t) \psi(t, x)$. With this choice, we have

$$\partial_t \Phi = \chi' \psi - M \chi Y'_\theta \quad \text{and} \quad \nabla \Phi = -\chi Y'_\theta \frac{x - x_0}{\|x - x_0\|}. \quad (4.7)$$

Setting $B(t, x, u, v) = |u - v|M + \text{sign}(u - v) (f(t, x, u) - f(t, x, v)) \cdot \frac{x - x_0}{\|x - x_0\|}$, the first line in (4.4) becomes

$$\begin{aligned} & \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \left[(u - v) \Psi \partial_t \Phi + (f(t, x, u) - f(t, x, v)) \cdot (\nabla \Phi) \Psi \right] \text{sign}(u - v) dx dt dy ds \\ &= \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} (|u - v| \chi' \psi - B(t, x, u, v) \chi Y'_\theta) \Psi dx dt dy ds \\ &\leq \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} |u - v| \chi' \psi \Psi dx dt dy ds, \end{aligned}$$

since $B(t, x, u, v)$ is positive for all $(t, x, u, v) \in \Omega \times \mathbb{R}$. Due to the above estimate and

to (4.4), we have

$$\begin{aligned} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} & \left[(u-v) \chi' \psi \Psi \right. \\ & + (f(s, y, v) - f(s, y, u) - f(t, x, v) + f(t, x, u)) \cdot (\nabla \Psi) \Phi \\ & + (F(t, x, u) - F(s, y, v) - \operatorname{div} f(t, x, v) + \operatorname{div} f(s, y, u)) \varphi \left. \right] \\ & \times \operatorname{sign}(u-v) dx dt dy ds \geq 0. \end{aligned}$$

Now, we aim at bounds for each term of this sum. Introduce the following notations:

$$\begin{aligned} I &= \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} |u-v| \chi' \psi \Psi dx dt dy ds, \\ J_x &= \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} (f(t, y, v) - f(t, y, u) + f(t, x, u) - f(t, x, v)) \cdot (\nabla \Psi) \Phi \\ & \quad \times \operatorname{sign}(u-v) dx dt dy ds, \end{aligned} \quad (4.8)$$

$$\begin{aligned} J_t &= \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} (f(s, y, v) - f(s, y, u) + f(t, y, u) - f(t, y, v)) \cdot (\nabla \Psi) \Phi \\ & \quad \times \operatorname{sign}(u-v) dx dt dy ds, \end{aligned} \quad (4.9)$$

$$\begin{aligned} L_1 &= \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} [\operatorname{div} f(t, x, v) - \operatorname{div} f(t, x, u) + F(t, y, v) - F(t, y, u)] \\ & \quad \varphi \operatorname{sign}(u-v) dx dt dy ds \end{aligned} \quad (4.10)$$

$$\begin{aligned} L_2 &= \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \left[\int_0^1 \nabla (F - \operatorname{div} f)(t, rx + (1-r)y, u) \cdot (x-y) dr \right] \varphi \\ & \quad \times \operatorname{sign}(u-v) dx dt dy ds. \end{aligned} \quad (4.11)$$

$$\begin{aligned} L_t &= \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} (F(t, y, v) - F(s, y, v) - \operatorname{div} f(t, y, u) + \operatorname{div} f(s, y, u)) \varphi \\ & \quad \times \operatorname{sign}(u-v) dx dt dy ds. \end{aligned} \quad (4.12)$$

Then, the above inequality is rewritten as $I + J_x + J_t + L_1 + L_2 + L_t \geq 0$. Choose $\Psi(t, x) = \nu(t) \mu(x)$ where, for $\eta, \lambda > 0$, $\mu \in \mathcal{C}_c^\infty(\mathbb{R}_+; \mathbb{R}_+)$ satisfies (6.1)–(6.2) and

$$\nu(t) = \frac{1}{\eta} \nu_1\left(\frac{t}{\eta}\right), \quad \int_{\mathbb{R}} \nu_1(s) ds = 1, \quad \nu_1 \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{R}_+), \quad \operatorname{supp}(\nu_1) \subset]-1, 0[. \quad (4.13)$$

Now, we want to estimate separately I , J_x , J_t , L_1 , L_2 and L_t . Note first that if $x, y \in \mathbb{R}^N \setminus \{\bigcup_{t \in [0, T_0]} \operatorname{Supp} u(t)\}$, the integrand in J_x and L_1 vanishes, so denoting

$$\mathcal{S}_T(u) = \bigcup_{t \in [0, T_0]} \operatorname{Supp} u(t), \quad (4.14)$$

the space of integration of J_x and L_1 is in fact $\mathbb{R}_+ \times \mathcal{S}_T(u) \times \mathbb{R}_+ \times \mathcal{S}_T(u)$. The main differences with respect to the proof of [7, Theorem 2.5] are the following:

- The \mathbf{L}^∞ norm that we took on $\mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}$, are now taken on $\Sigma_{T_0}^u = [0, T_0] \times \mathcal{S}_T(u) \times [-\mathcal{U}, \mathcal{U}]$, where $\mathcal{U} = \sup(\|u(t)\|_{\mathbf{L}^\infty(\mathbb{R}^N)}, t \in [0, T_0])$.

- For J_t and L_t , by Dominated Convergence Theorem, we get when $\eta \rightarrow 0$

$$\lim_{\eta \rightarrow 0} J_t = \lim_{\eta \rightarrow 0} L_t = 0, \quad (4.15)$$

which avoids the use of time derivatives.

- The \mathbf{L}^∞ norm of u in L_2 is now taken on $[-U_t, U_t]$ where $U_t = \|u(t)\|_{\mathbf{L}^\infty(\mathbb{R}^N)}$.

We do not rewrite the estimates on I , J_x , L_1 , L_2 , that are the same as in [7, Theorem 2.5], up to the space in the \mathbf{L}^∞ norm. See remark 4.1 for precisions on the estimate of L_2 .

Letting $\varepsilon, \eta, \theta \rightarrow 0$ we get

$$\begin{aligned} \limsup_{\varepsilon, \eta, \theta \rightarrow 0} I &= \int_{\mathbb{R}^N} \int_{\|x-x_0\| \leq R+MT_0} |u(0, x) - u(0, y)| \mu(x-y) \, dx \, dy \\ &\quad - \int_{\mathbb{R}^N} \int_{\|x-x_0\| \leq R+M(T_0-T)} |u(T, x) - u(T, y)| \mu(x-y) \, dx \, dy, \\ \limsup_{\varepsilon, \eta, \theta \rightarrow 0} J_x &\leq \|\nabla \partial_u f\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} \int_0^T \int_{\mathbb{R}^N} \int_{B(x_0, R+M(T_0-t))} \|x-y\| |u(t, x) - u(t, y)| \\ &\quad \times \|\nabla \mu(x-y)\| \, dx \, dy \, dt, \\ \limsup_{\varepsilon, \eta, \theta \rightarrow 0} J_t &= 0, \\ \limsup_{\varepsilon, \eta, \theta \rightarrow 0} L_1 &\leq \left(N \|\nabla \partial_u f\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} \right) \\ &\quad \times \int_0^T \int_{\mathbb{R}^N} \int_{\|x-x_0\| \leq R+M(T_0-t)} |u(t, x) - u(t, y)| \mu(x-y) \, dx \, dy \, dt, \\ \limsup_{\varepsilon, \eta, \theta \rightarrow 0} L_2 &= \lambda M_1 \int_0^T \int_{\mathbb{R}^N} \|\nabla(F - \operatorname{div} f)(t, y, \cdot)\|_{\mathbf{L}^\infty([-U_t, U_t])} \, dy \, dt, \\ \limsup_{\varepsilon, \eta, \theta \rightarrow 0} L_t &= 0, \end{aligned}$$

where

$$M_1 = \int_{\mathbb{R}^N} \|x\| \mu_1(\|x\|) \, dx. \quad (4.16)$$

Above, the right hand sides are bounded thanks to **(H2*)**.

Collating all the obtained results and using the equality, $\|\nabla\mu(x)\| = -\frac{1}{\lambda^{N+1}}\mu'_1\left(\frac{\|x\|}{\lambda}\right)$

$$\begin{aligned}
& \int_{\mathbb{R}^N} \int_{\|x-x_0\| \leq R+M(T_0-T)} |u(T, x) - u(T, y)| \frac{1}{\lambda^N} \mu_1\left(\frac{\|x-y\|}{\lambda}\right) dx dy \\
\leq & \int_{\mathbb{R}^N} \int_{\|x-x_0\| \leq R+MT_0} |u(0, x) - u(0, y)| \frac{1}{\lambda^N} \mu_1\left(\frac{\|x-y\|}{\lambda}\right) dx dy \\
& - \|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} \int_0^T \int_{\mathbb{R}^N} \int_{\|x-x_0\| \leq R+M(T_0-t)} |u(t, x) - u(t, y)| \\
& \quad \times \frac{1}{\lambda^{N+1}} \mu'_1\left(\frac{\|x-y\|}{\lambda}\right) \|x-y\| dx dy dt \\
& + \left(N\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)}\right) \int_0^T \int_{\mathbb{R}^N} \int_{\|x-x_0\| \leq R+M(T_0-t)} |u(t, x) - u(t, y)| \\
& \quad \times \frac{1}{\lambda^N} \mu_1\left(\frac{\|x-y\|}{\lambda}\right) dx dy dt \\
& + \lambda M_1 \int_0^T \int_{\mathbb{R}^N} \|\nabla(F - \operatorname{div} f)(t, y, \cdot)\|_{\mathbf{L}^\infty([-U_t, U_t])} dy dt.
\end{aligned} \tag{4.17}$$

If $\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} = \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} = 0$ and under the present assumption that $u_0 \in \mathcal{C}^1(\mathbb{R}^N; \mathbb{R})$, using Proposition 3.2, (3.4) and (4.16), we directly obtain that

$$\operatorname{TV}(u(T)) \leq \operatorname{TV}(u_0) + \frac{M_1}{C_1} \int_0^T \int_{\mathbb{R}^N} \|\nabla(F - \operatorname{div} f)(t, y, \cdot)\|_{\mathbf{L}^\infty([-U_t, U_t])} dy dt. \tag{4.18}$$

The same procedure at the end of this proof allows to extend (4.18) to more general initial data, providing an estimate of $\operatorname{TV}(u(t))$ in the situation studied in [2].

Now, it remains to treat the case when $\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} \neq 0$. As in [7, Theorem 2.5], a direct use of Gronwall lemma is not possible, but we can first obtain an estimate of the function:

$$\mathcal{F}(T, \lambda) = \int_0^T \int_{\mathbb{R}^N} \int_{\|x-x_0\| \leq R+M(T_0-t)} |u(t, x) - u(t, x-z)| \frac{1}{\lambda^N} \mu_1\left(\frac{\|z\|}{\lambda}\right) dx dz dt.$$

Indeed, we get that if T is such that

$$T < \frac{1}{(1+2N)\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)}},$$

then we obtain, with $\alpha = \left(2N\|\nabla\partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty} - \frac{1}{T}\right) \left(\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)}\right)^{-1} < -1$,

$$\frac{1}{\lambda} \mathcal{F}(T', \lambda) \leq \frac{1}{-\alpha-1} (M_1 \operatorname{TV}(u_0) + C(T')) \frac{1}{\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)}}. \tag{4.19}$$

Furthermore, by (6.1) and (6.2) there exists a constant $Q > 0$ such that for all $z \in \mathbb{R}^N$

$$-\mu'_1(\|z\|) \leq Q \mu_1\left(\frac{\|z\|}{2}\right). \tag{4.20}$$

Divide both sides in (4.17) by λ , rewrite them using (4.19), (4.20), apply (3.1) and obtain

$$\begin{aligned} & \frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{\|x-x_0\| \leq R+M(T_0-T)} |u(T, x) - u(T, y)| \frac{1}{\lambda^N} \mu_1 \left(\frac{\|x-y\|}{\lambda} \right) dx dy \\ & \leq M_1 \text{TV}(u_0) + \frac{\mathcal{F}(T, \lambda)}{\lambda} \left(2N \|\nabla \partial_u f\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} \right) \\ & \quad + \frac{\mathcal{F}(T, 2\lambda)}{2\lambda} 2^{N+2} Q \|\nabla \partial_u f\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} \\ & \quad + M_1 \int_0^T \int_{\mathbb{R}^N} \|\nabla(F - \text{div } f)(t, y, \cdot)\|_{\mathbf{L}^\infty([-U_t, U_t])} dy dt. \end{aligned}$$

An application of (4.19) yields an estimate of the type

$$\frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{B(x_0, R+M(T_0-T))} |u(T, x) - u(T, x-z)| \mu(z) dx dz \leq \check{C}, \quad (4.21)$$

where the positive constant \check{C} is independent from R and λ . Applying Proposition 3.2 we obtain that $u(t) \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$ for $t \in [0, 2T_1[$, where

$$T_1 = \frac{1}{2 \left((1+2N) \|\nabla \partial_u f\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} \right)}. \quad (4.22)$$

The next step is to obtain a general estimate of the TV norm. The starting point is (4.17). Recall the definitions (4.16) of M_1 and (4.22) of T_1 . Moreover, by integration by part we obtain

$$\int_{\mathbb{R}^N} |z_1| \|z\| \mu'_1(\|z\|) dz = -(N+1) C_1.$$

The following step is *not* similar to [7, proof of theorem 2.5]: we divide both terms in (4.17) by λ , apply (3.3) on the first, second and third terms in the right hand side, with $\rho_1 = \mu_1 \geq 0$ in the second and third case, and with $\rho_1 = -\mu'_1 \geq 0$ in the second case. We obtain for all $T \in [0, T_1]$ with $T_1 < T_0$

$$\begin{aligned} \text{TV}(u(T)) & \leq \text{TV}(u_0) + \left((2N+1) \|\nabla \partial_u f\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} \right) \int_0^T \text{TV}(u(t)) dt \\ & \quad + \frac{M_1}{C_1} \int_0^T \int_{\mathbb{R}^N} \|\nabla(F - \text{div } f)(t, x, \cdot)\|_{\mathbf{L}^\infty([-U_t, U_t])} dx dt. \end{aligned}$$

Next, an application of the Gronwall Lemma shows that $\text{TV}(u(t))$ is bounded on $[0, T_1]$

$$\text{TV}(u(T)) \leq e^{\kappa_0^* T} \text{TV}(u_0) + \frac{M_1}{C_1} \int_0^T e^{\kappa_0^*(T-t)} \int_{\mathbb{R}^N} \|\nabla(F - \text{div } f)(t, x, \cdot)\|_{\mathbf{L}^\infty([-U_t, U_t])} dx dt \quad (4.23)$$

for $T \in [0, T_1]$, M_1, C_1 as in (4.16), (3.4) and $\kappa_0^* = (2N+1) \|\nabla \partial_u f\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)}$.

Now, it remains only to relax assumption on the regularity of u_0 and to note that the bound (4.23) is additive in time. These steps are the same as in [7, Theorem 2.5], so we do not write them. \square

Remark 4.1. The constant NW_N in front of $\int_0^T \int_{\mathbb{R}^N} \|\nabla(F - \operatorname{div} f)(t, x, \cdot)\|_{\mathbf{L}^\infty([-U_t, U_t])} dx dt$ in Theorem 2.2 comes from the estimate of the term L_2 defined by (4.11).

We have indeed

$$\begin{aligned}
L_2 &\leq \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_0^1 |\nabla(F - \operatorname{div} f)(t, x - \lambda(1-r)z, u) \cdot (\lambda z)| \\
&\quad \times \chi \psi \mu_1(\|z\|) \nu dr dx dt dz ds \\
&\leq \lambda \int_0^{T+\varepsilon} \int_{B(x_0, R+M(T_0-t)+\theta)} \int_{\mathbb{R}^N} \int_0^1 |\nabla(F - \operatorname{div} f)(t, x - \lambda(1-r)z, u) \cdot (z)| \\
&\quad \times \mu_1(\|z\|) dr dz dx dt \\
&\leq \lambda \int_0^{T+\varepsilon} \int_0^1 \int_{B(x_0, R+M(T_0-t)+\theta+\lambda)} \int_{\mathbb{R}^N} |\nabla(F - \operatorname{div} f)(t, x', u(t, x' + \lambda(1-r)z)) \cdot z| \\
&\quad \times \mu_1(\|z\|) dz dx' dr dt
\end{aligned}$$

If $F - \operatorname{div} f$ does not depend on u , then, with the same computations as in the proof of Proposition 3.2, considering $z \mapsto \nabla(F - \operatorname{div} f)(t, x') \cdot z$ as a linear application, we get:

$$L_2 \leq \lambda \int_0^{T+\varepsilon} \int_{B(x_0, R+M(T_0-t)+\theta+\lambda)} |\nabla(F - \operatorname{div} f)(t, x')| dx' dt \int_{\mathbb{R}^N} |z| \mu_1(\|z\|) dz,$$

which allows us to get rid of the constant NW_N into the bound of L_2 .

However, in the general case, because of the dependence of u in z , we are led to take the supremum of $u(t)$. We obtain the following:

$$L_2 \leq \lambda \int_0^{T+\varepsilon} \int_{B(x_0, R+M(T_0-t)+\theta+\lambda)} \int_{\mathbb{R}^N} \sup_{y \in \mathbb{R}^N} |\nabla(F - \operatorname{div} f)(t, x', u(t, y)) \cdot z| \mu_1(\|z\|) dz dx' dt.$$

We can no longer do the same computations as in the proof of Proposition 3.2. Indeed, it is not allowed to permute sup and $\int_{\mathbb{R}^N}$, consequently, if we want to isolate the variable z from the other variables, we use the Cauchy-Schwartz inequality to obtain:

$$\begin{aligned}
L_2 &\leq \lambda \int_0^{T+\varepsilon} \int_{B(x_0, R+M(T_0-t)+\theta+\lambda)} \sup_{y \in \mathbb{R}^N} \|\nabla(F - \operatorname{div} f)(t, x', u(t, y))\| dx' dt \\
&\quad \times \int_{\mathbb{R}^N} \|z\| \mu_1(\|z\|) dz.
\end{aligned}$$

The constant NW_N appears here when we divide by $C_1 = \int_{\mathbb{R}^N} |z| \mu_1(\|z\|) dz$, since, by Lemma 6.1, $\frac{1}{C_1} \int_{\mathbb{R}^N} \|z\| \mu_1(\|z\|) dz = NW_N$.

In the general case, we were consequently not able, using this method, to erase the constant NW_N on the right hand side of (2.4).

5 Proof of Theorem 2.4.

We give now the proof of Theorem 2.4. The beginning of this proof is similar, up to (5.4), to the proof of Theorem 2.6 in [7]. We rewrite it in order to be complete and clear.

Proof of Theorem 2.4. Let $\Phi \in \mathcal{C}_c^\infty(\mathbb{R}_+^* \times \mathbb{R}^N; \mathbb{R}_+)$, $\Psi \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{R}^N; \mathbb{R}_+)$, and set $\varphi(t, x, s, y) = \Phi(t, x) \Psi(t - s, x - y)$ as in (4.3).

By Kruřkov Theorem [10, Theorem 5 & Section 5, Remark 4], the set of hypotheses **(H1*)** gives us existence and uniqueness of a weak entropy solution for any initial condition in $\mathbf{L}^\infty \cap \mathbf{L}^1(\mathbb{R}^N; \mathbb{R})$. Let u be the Kruřkov solution associated to u_0 and v be the Kruřkov solution associated to v_0 . By definition of Kruřkov weak entropy solution, we have for all $l \in \mathbb{R}$, for all $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}^N$

$$\int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \left[(u - l) \partial_s \varphi + (f(s, y, u) - f(s, y, l)) \cdot \nabla_y \varphi + (F(s, y, u) - \operatorname{div} f(s, y, l)) \varphi \right] \times \operatorname{sign}(u - l) \, dy \, ds \geq 0 \quad (5.1)$$

and for all $k \in \mathbb{R}$, for all $(s, y) \in \mathbb{R}_+^* \times \mathbb{R}^N$

$$\int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \left[(v - k) \partial_t \varphi + (g(t, x, v) - g(t, x, k)) \cdot \nabla_x \varphi + (G(t, x, v) - \operatorname{div} g(t, x, k)) \varphi \right] \times \operatorname{sign}(v - k) \, dx \, dt \geq 0. \quad (5.2)$$

Choose $k = u(s, y)$ in (5.2) and integrate with respect to (s, y) . Analogously, take $l = v(t, x)$ in (5.1) and integrate with respect to (t, x) . By summing the obtained equations, we get, denoting $u = u(s, y)$ and $v = v(t, x)$:

$$\begin{aligned} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} & \left[(u - v) \Psi \partial_t \Phi + (g(t, x, u) - g(t, x, v)) \cdot (\nabla \Phi) \Psi \right. \\ & + (g(t, x, u) - g(t, x, v) - f(s, y, u) + f(s, y, v)) \cdot (\nabla \Psi) \Phi \\ & + (F(s, y, u) - G(t, x, v) + \operatorname{div} g(t, x, u) - \operatorname{div} f(s, y, v)) \varphi \left. \right] \\ & \times \operatorname{sign}(u - v) \, dx \, dt \, dy \, ds \geq 0. \end{aligned} \quad (5.3)$$

We introduce a family of functions $\{Y_\vartheta\}_{\vartheta > 0}$ as in (4.5). Let $T_0 > 0$ and denote $M = \|\partial_u g\|_{\mathbf{L}^\infty(\Omega_{T_0}^\vee; \mathbb{R}^N)}$ with $\mathcal{V} = \max(\|u\|_{\mathbf{L}^\infty([0, T_0] \times \mathbb{R}^N)}, \|v\|_{\mathbf{L}^\infty([0, T_0] \times \mathbb{R}^N)})$. We also define χ, ψ as in (4.6), for $\varepsilon, \theta, R > 0$, $x_0 \in \mathbb{R}^N$ (see also Figure 2). Note that with these choices, equalities (4.7) still hold. Note that here the definition of the test function φ is essentially the same as in the preceding proof; the only change is the definition of the constant M , which is now defined with reference to g . We also introduce as above the function $B(t, x, u, v) = M|u - v| + \operatorname{sign}(u - v) (g(t, x, u) - g(t, x, v)) \cdot \frac{x - x_0}{\|x - x_0\|}$ that is positive for all $(t, x, u, v) \in \Omega \times \mathbb{R}$, and we have:

$$\begin{aligned} & \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \left[(u - v) \partial_t \Phi + (g(t, x, u) - g(t, x, v)) \cdot \nabla \Phi \right] \Psi \operatorname{sign}(u - v) \, dx \, dt \, dy \, ds \\ & \leq \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} [|u - v| \chi' \psi - B(t, x, u, v) \chi Y_\theta'] \Psi \, dx \, dt \, dy \, ds \\ & \leq \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} |u - v| \chi' \psi \Psi \, dx \, dt \, dy \, ds. \end{aligned}$$

Due to the above estimate and (5.3), we obtain

$$\begin{aligned} & \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \left[(u-v)\chi'\psi\Psi \right. \\ & \quad + (g(t,x,u) - g(t,x,v) - f(s,y,u) + f(s,y,v)) \cdot (\nabla\Psi)\Phi \\ & \quad + (F(s,y,u) - G(t,x,v) + \operatorname{div} g(t,x,u) - \operatorname{div} f(s,y,v)) \varphi \left. \right] \\ & \quad \times \operatorname{sign}(u-v) \, dx \, dt \, dy \, ds \geq 0, \end{aligned}$$

i.e. $I + J_x + J_t + K + L_x + L_t \geq 0$, where

$$I = \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} |u-v|\chi'\psi\Psi \, dx \, dt \, dy \, ds, \quad (5.4)$$

$$\begin{aligned} J_x = & \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \left[(g(t,x,u) - g(t,x,v) + g(t,y,v) - g(t,y,u)) \cdot (\nabla\Psi)\Phi \right. \\ & \quad \left. + (\operatorname{div} g(t,x,u) - \operatorname{div} g(t,x,v))\varphi \right] \operatorname{sign}(u-v) \, dx \, dt \, dy \, ds, \end{aligned} \quad (5.5)$$

$$\begin{aligned} J_t = & \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \left[(f(s,y,v) - f(s,y,u) + f(t,y,u) - f(t,y,v)) \cdot (\nabla\Psi)\Phi \right. \\ & \quad \left. + (\operatorname{div} f(t,y,v) - \operatorname{div} f(s,y,v)) \right] \times \operatorname{sign}(u-v) \, dx \, dt \, dy \, ds, \end{aligned}$$

$$\begin{aligned} K = & \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} ((g-f)(t,y,u) - (g-f)(t,y,v)) \cdot (\nabla\Psi)\Phi \\ & \quad \times \operatorname{sign}(u-v) \, dx \, dt \, dy \, ds, \end{aligned} \quad (5.6)$$

$$\begin{aligned} L_x = & \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} (F(t,y,u) - G(t,x,v) + \operatorname{div} g(t,x,v) - \operatorname{div} f(t,y,v)) \varphi \\ & \quad \times \operatorname{sign}(u-v) \, dx \, dt \, dy \, ds, \end{aligned} \quad (5.7)$$

$$\begin{aligned} L_t = & \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} (F(s,y,u) - F(t,y,u)) \varphi \\ & \quad \times \operatorname{sign}(u-v) \, dx \, dt \, dy \, ds. \end{aligned}$$

Now, we choose $\Psi(t,x) = \nu(t)\mu(x)$ as in (4.13), (6.1), (6.2). The estimate on I is the same as in the proof of [7, Theorem 2.6]: thanks to Lemma 6.2, we obtain

$$\begin{aligned} \limsup_{\varepsilon, \eta, \lambda \rightarrow 0} I & \leq \int_{\|x-x_0\| \leq R+MT_0+\theta} |u(0,x) - v(0,x)| \, dx \\ & \quad - \int_{\|x-x_0\| \leq R+M(T_0-T)} |u(T,x) - v(T,x)| \, dx. \end{aligned} \quad (5.8)$$

However, for J_x , we derive a new estimate with respect to [7, Theorem 2.6]. Indeed, as g is \mathcal{C}^2 in space, we can use the following Taylor expansion:

$$\begin{aligned} g(t,y,v) &= g(t,x,v) + \nabla g(t,x,v) \cdot (y-x) \\ & \quad + \int_0^1 (1-r) \nabla^2 g(t, ry + (1-r)x, v) \, dr \cdot (y-x)^2, \\ g(t,y,u) &= g(t,x,u) + \nabla g(t,x,u) \cdot (y-x) \\ & \quad + \int_0^1 (1-r) \nabla^2 g(t, ry + (1-r)x, u) \, dr \cdot (y-x)^2. \end{aligned}$$

Besides, we note that

$$\begin{aligned}
& (\nabla g(t, x, v) \cdot (y - x)) \cdot \nabla \mu(x - y) - \operatorname{div} g(t, x, v) \mu(x - y) \\
&= \sum_{i,j} \partial_j g_i(t, x, v) (y_j - x_j) \partial_i \mu(x - y) - \sum_i \partial_i g_i(t, x, v) \mu(x - y) \\
&= - \sum_{i,j} \partial_j g_i(t, x, v) \partial_i (z_j \mu(z)) \big|_{z=x-y} \\
&= - \nabla g(t, x, v) \cdot \nabla((x - y) \mu(x - y))
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
& (\nabla g(t, x, u) \cdot (x - y)) \nabla \mu(x - y) + \operatorname{div} g(t, x, u) \mu(x - y) \\
&= \nabla g(t, x, u) \cdot \nabla((x - y) \mu)
\end{aligned}$$

so that finally

$$\begin{aligned}
& (g(t, y, v) - g(t, x, v) + g(t, x, u) - g(t, y, u)) \nabla \mu + (\operatorname{div} g(t, x, u) - \operatorname{div} g(t, x, v)) \mu(x - y) \\
&= (\nabla g(t, x, u) - \nabla g(t, x, v)) \cdot \nabla((x - y) \mu) \\
&+ \left[\int_0^1 (1 - r) \left(\nabla^2 g(t, ry + (1 - r)x, u) - \nabla^2 g(t, ry + (1 - r)x, v) \right) dr \cdot (x - y)^2 \right] \cdot \nabla \mu
\end{aligned}$$

After a change of variable, we obtain

$$\begin{aligned}
& \lim_{\varepsilon, \eta, \theta \rightarrow 0} J_x \\
&= \int_0^T \int_{B(x_0, R+M(T_0-t))} \int_{\mathbb{R}^N} \left\{ \left(\nabla g(t, x, u(t, x - \lambda z)) - \nabla g(t, x, v(t, x)) \right) \right. \\
&\quad \cdot \nabla(z \mu_1(\|z\|)) \operatorname{sign}(u - v) \\
&\quad + \lambda \left[\int_0^1 (1 - r) \left(\nabla^2 g(t, ry + (1 - r)x, u) - \nabla^2 g(t, ry + (1 - r)x, v) \right) dr \cdot z^2 \right] \\
&\quad \cdot \frac{z}{\|z\|} \mu'_1(\|z\|) \operatorname{sign}(u - v) \Big\} dz dx dt .
\end{aligned}$$

When λ goes to 0, we obtain by the Dominated Convergence Theorem

$$\begin{aligned}
\lim_{\varepsilon, \eta, \theta, \lambda \rightarrow 0} J_x &= \int_0^T \int_{B(x_0, R+M(T_0-t))} \left(\nabla g(t, x, u(t, x)) - \nabla g(t, x, v(t, x)) \right) \operatorname{sign}(u - v) dx dt \\
&\quad \cdot \int_{\mathbb{R}^N} \nabla(z \mu_1(\|z\|)) dz .
\end{aligned}$$

As $\int_{\mathbb{R}^N} \nabla(z \mu_1(\|z\|)) dz = 0$, we finally get

$$\lim_{\varepsilon, \eta, \theta, \lambda \rightarrow 0} J_x = 0 . \tag{5.9}$$

For J_t and L_t , we avoid now the use of the derivatives in time thanks to an application of the Dominated Convergence Theorem. We obtain

$$\lim_{\varepsilon, \eta, \theta, \lambda \rightarrow 0} J_t = \lim_{\varepsilon, \eta, \theta, \lambda \rightarrow 0} L_t = 0 . \tag{5.10}$$

For L_x , we have

$$L_x \leq \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \left[(F - G - \operatorname{div}(f - g))(t, y, v) + (F(t, y, u) - F(t, y, v)) \right. \\ \left. + \int_0^1 \nabla G(t, ry + (1-r)x, v) \cdot (y - x) dr \right] \varphi dy ds dx dt .$$

Note that $F(t, y, u) - F(t, y, v) = \int_v^u \partial_u F(t, y, w) dw$ vanishes for $y \in \mathbb{R}^N \setminus \mathcal{S}_T(u, v)$. Consequently, with $\mathcal{V} = \sup_{t \in [0, T_0]} (\|u(t)\|_{\mathbf{L}^\infty(\mathbb{R}^N)}, \|v(t)\|_{\mathbf{L}^\infty(\mathbb{R}^N)})$ and

$$\Sigma_{T_0}^{u,v} = [0, T_0] \times \mathcal{S}_T(u, v) \times [-\mathcal{V}, \mathcal{V}], \quad (5.11)$$

we obtain

$$\lim_{\varepsilon, \eta, \theta, \lambda \rightarrow 0} L_x \leq \int_0^T \int_{B(x_0, R+M(T_0-t))} \left\| (F - G)(t, x, \cdot) - \operatorname{div}(f - g)(t, x, \cdot) \right\|_{\mathbf{L}^\infty([-U_t, U_t])} dx dt \\ + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma_{T_0}^{u,v})} \int_0^T \int_{B(x_0, R+M(T_0-t))} |u(t, x) - v(t, x)| dx dt . \quad (5.12)$$

In order to estimate K as given in (5.6), we follow the same procedure as in [7, Theorem 2.6]: let us introduce a regularisation of the y dependent functions. In fact, let $\rho_\alpha(z) = \frac{1}{\alpha} \rho\left(\frac{z}{\alpha}\right)$ and $\sigma_\beta(y) = \frac{1}{\beta^N} \sigma\left(\frac{y}{\beta}\right)$, where $\rho \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{R}_+)$ and $\sigma \in \mathcal{C}_c^\infty(\mathbb{R}^N; \mathbb{R}_+)$ are such that $\|\rho\|_{\mathbf{L}^1(\mathbb{R}; \mathbb{R})} = \|\sigma\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} = 1$ and $\operatorname{Supp}(\rho) \subseteq]-1, 1[$, $\operatorname{Supp}(\sigma) \subseteq B(0, 1)$. Then, we introduce

$$\begin{aligned} P(w) &= (g - f)(t, y, w), & s_\alpha &= \operatorname{sign} *_u \rho_\alpha, \\ \Upsilon_\alpha^i(w) &= s_\alpha(w - v) (P_i(w) - P_i(v)), & u_\beta &= \sigma_\beta *_y u, \\ \Upsilon^i(w) &= \operatorname{sign}(w - v) (P_i(w) - P_i(v)), \end{aligned}$$

so that we obtain

$$\begin{aligned} &\langle \Upsilon_\alpha^i(u_\beta) - \Upsilon_\alpha^i(u), \partial_{y_i} \varphi \rangle \\ &= \int_{\mathbb{R}^N} \left[(s_\alpha(u - v) P_i(u) - s_\alpha(u_\beta - v) P_i(u_\beta)) + (s_\alpha(u - v) - s_\alpha(u_\beta - v)) P_i(v) \right] \partial_{y_i} \varphi dy \\ &= \int_{\mathbb{R}^N} \int_u^{u_\beta} (\partial_U (s_\alpha(U - v) P_i(U)) - \partial_U s_\alpha(U - v) P_i(v)) \partial_{y_i} \varphi dy \\ &= \int_{\mathbb{R}^N} \int_u^{u_\beta} (s'_\alpha(U - v) (P_i(U) - P_i(v)) + s_\alpha(U - v) P'_i(U)) \partial_{y_i} \varphi dy . \end{aligned}$$

Now, we use the relation $s'_\alpha(U) = \frac{2}{\alpha} \rho\left(\frac{U}{\alpha}\right)$ to obtain

$$\begin{aligned} \left| \langle \Upsilon_\alpha^i(u_\beta) - \Upsilon_\alpha^i(u), \partial_{y_i} \varphi \rangle \right| &\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}} 2\rho(z) |P_i(v + \alpha z) - P_i(v)| \partial_{y_i} \varphi dz dy \\ &\quad + \int_{\mathbb{R}^N} \int_u^{u_\beta} |P'_i(U)| \partial_{y_i} \varphi dU dy . \end{aligned}$$

When α tends to 0, using the Dominated Convergence Theorem we obtain

$$\left| \langle \Upsilon^i(u_\beta) - \Upsilon^i(u), \partial_{y_i} \varphi \rangle \right| \leq \int_{\mathbb{R}^N} |u - u_\beta| \|P'_i\|_{\mathbf{L}^\infty} \partial_{y_i} \varphi dy .$$

Applying the Dominated Convergence Theorem again, we see that

$$\begin{aligned}\lim_{\beta \rightarrow 0} \lim_{\alpha \rightarrow 0} \langle \Upsilon_\alpha^i(u_\beta), \partial_{y_i} \varphi \rangle &= \langle \Upsilon^i(u), \partial_{y_i} \varphi \rangle, \\ \lim_{\beta \rightarrow 0} \lim_{\alpha \rightarrow 0} \langle \Upsilon_\alpha(u_\beta), \nabla_y \varphi \rangle &= \langle \Upsilon(u), \nabla_y \varphi \rangle.\end{aligned}$$

Consequently, it is sufficient to find a bound independent of α and β on $K_{\alpha,\beta}$, where

$$K_{\alpha,\beta} = - \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \Upsilon_\alpha(u_\beta) \cdot \nabla_y \varphi \, dx \, dt \, dy \, ds.$$

Integrating by parts, we obtain

$$\begin{aligned}K_{\alpha,\beta} &= \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \operatorname{Div}_y \Upsilon_\alpha(u_\beta) \varphi \, dx \, dt \, dy \, ds \\ &= \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \partial_u s_\alpha(u_\beta - v) \nabla u_\beta \cdot ((g - f)(t, y, u_\beta) - (g - f)(t, y, v)) \varphi \, dx \, dt \, dy \, ds \\ &\quad + \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} s_\alpha(u_\beta - v) (\partial_u (g - f)(t, y, u_\beta) \cdot \nabla u_\beta) \varphi \, dx \, dt \, dy \, ds \\ &\quad + \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} s_\alpha(u_\beta - v) (\operatorname{div} (g - f)(t, y, u_\beta) - \operatorname{div} (g - f)(t, y, v)) \varphi \, dx \, dt \, dy \, ds \\ &= K_1 + K_2 + K_3.\end{aligned}$$

We now search for a bound for each term of the sum above.

- For K_1 , recall that $\partial_u s_\alpha(u) = \frac{2}{\alpha} \rho\left(\frac{u}{\alpha}\right)$. Hence, by the Dominated Convergence Theorem, we get that $K_1 \rightarrow 0$ when $\alpha \rightarrow 0$. Indeed,

$$\begin{aligned}& \left| \frac{2}{\alpha} \rho\left(\frac{u_\beta - v}{\alpha}\right) \nabla u_\beta \cdot ((g - f)(t, y, u_\beta) - (g - f)(t, y, v)) \varphi \right| \\ & \leq \frac{2}{\alpha} \rho\left(\frac{u_\beta - v}{\alpha}\right) \varphi \|\nabla u_\beta(s, y)\| \int_v^{u_\beta} \|\partial_u (f - g)(t, y, w)\| \, dw \\ & \leq 2 \|\rho\|_{\mathbf{L}^\infty(\mathbb{R}; \mathbb{R})} \|\nabla u_\beta(s, y)\| \|\partial_u (f - g)\|_{\mathbf{L}^\infty(\Omega_U^{T_0}; \mathbb{R}^N)} \varphi \in \mathbf{L}^1\left((\mathbb{R}_+^* \times \mathbb{R}^N)^2; \mathbb{R}\right).\end{aligned}$$

- For K_2 , denoting $\mathcal{D} = [0, T_0] \times \{\mathcal{S}_{T_0}(u) + B(0, \beta)\} \times [-\|u\|_{\mathbf{L}^\infty}, \|u\|_{\mathbf{L}^\infty}]$, we get

$$\begin{aligned}K_2 &\leq \|\partial_u (f - g)\|_{\mathbf{L}^\infty(\mathcal{D}; \mathbb{R}^N)} \int_0^{T+\varepsilon+\eta} \int_{\mathbb{R}^N} \|\nabla u_\beta(s, y)\| \, dy \, ds \\ &\leq \|\partial_u (f - g)\|_{\mathbf{L}^\infty(\mathcal{D}; \mathbb{R}^N)} \int_0^{T+\varepsilon+\eta} \operatorname{TV}(u_\beta(t)) \, dt.\end{aligned}$$

We note besides that $\mathcal{D} \rightarrow \Sigma_{T_0}^u$ when $\beta \rightarrow 0$.

- For K_3 , we just pass to the limit in α, β and then make λ goes to 0. We see that:

$$K_3 \rightarrow_{\alpha, \beta, \lambda \rightarrow 0} 0.$$

Finally, letting $\alpha, \beta \rightarrow 0$ and $\varepsilon, \eta, \lambda \rightarrow 0$, due to [1, Proposition 3.7], we obtain

$$\limsup_{\varepsilon, \eta, \lambda \rightarrow 0} K \leq \|\partial_u(f - g)\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u; \mathbb{R}^N)} \int_0^T \text{TV}(u(t)) \, dt. \quad (5.13)$$

Now, we collate the estimates obtained in (5.8), (5.9), (5.12), and (5.13). Remark the order in which we pass to the various limits: first $\varepsilon, \eta, \theta \rightarrow 0$ and, after, $\lambda \rightarrow 0$. Therefore, we get

$$\begin{aligned} & \int_{B(x_0, R+M(T_0-T))} |u(T, x) - v(T, x)| \, dx \\ & \leq \int_{B(x_0, R+MT_0)} |u(0, x) - v(0, x)| \, dx \\ & \quad + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma_{T_0}^{u,v})} \int_0^T \int_{B(x_0, R+M(T_0-t))} |v(t, x) - u(t, x)| \, dx \, dt \\ & \quad + \left[\|\partial_u(f - g)\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} \int_0^T \text{TV}(u(t)) \, dt \right. \\ & \quad \left. + \int_0^T \int_{B(x_0, R+M(T_0-t))} \left\| ((F - G) - \text{div}(f - g))(t, y, \cdot) \right\|_{\mathbf{L}^\infty([-V_t, V_t])} \, dy \, dt \right] \end{aligned}$$

or equivalently

$$A'(T) \leq A'(0) + \kappa^* A(T) + S(T), \quad (5.14)$$

where

$$\begin{aligned} A(T) &= \int_0^T \int_{B(x_0, R+M(T_0-t))} |v(t, x) - u(t, x)| \, dx \, dt, \\ \kappa^* &= \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma_{T_0}^{u,v})}, \\ S(T) &= \|\partial_u(f - g)\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} \int_0^T \text{TV}(u(t)) \, dt \\ & \quad + \int_0^T \int_{B(x_0, R+M(T_0-t))} \left\| ((F - G) - \text{div}(f - g))(t, y, \cdot) \right\|_{\mathbf{L}^\infty([-V_t, V_t])} \, dy \, dt. \end{aligned} \quad (5.15)$$

The bound (2.4) on $\text{TV}(u(t))$ gives:

$$S(T) \leq \frac{e^{\kappa_0^* T} - 1}{\kappa_0^*} a + \int_0^T \frac{e^{\kappa_0^*(T-t)} - 1}{\kappa_0^*} b(t) \, dt + \int_0^T c(t) \, dt,$$

where κ_0^* is defined in (2.3) and

$$\begin{aligned} a &= \|\partial_u(f - g)\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} \text{TV}(u_0), \\ b(t) &= NW_N \|\partial_u(f - g)\|_{\mathbf{L}^\infty(\Sigma_{T_0}^u)} \int_{\mathbb{R}^N} \|\nabla(F - \text{div} f)(t, x, \cdot)\|_{\mathbf{L}^\infty([-U_t, U_t])} \, dx, \\ c(t) &= \int_{B(x_0, R+M(T_0-t))} \left\| ((F - G) - \text{div}(f - g))(t, y, \cdot) \right\|_{\mathbf{L}^\infty([-V_t, V_t])} \, dy, \end{aligned}$$

since $T \leq T_0$. Consequently

$$A'(T) \leq A'(0) + \kappa^* A(T) + \left(\frac{e^{\kappa_0^* T} - 1}{\kappa_0^*} a + \int_0^T \frac{e^{\kappa_0^* (T-t)} - 1}{\kappa_0^*} b(t) dt + \int_0^T c(t) dt \right). \quad (5.16)$$

By a Gronwall type argument, we obtain

$$A'(T) \leq e^{\kappa^* T} A'(0) + \frac{e^{\kappa_0^* T} - e^{\kappa^* T}}{\kappa_0^* - \kappa^*} a + \int_0^T \frac{e^{\kappa_0^* (T-t)} - e^{\kappa^* (T-t)}}{\kappa_0^* - \kappa^*} b(t) dt + \int_0^T e^{\kappa^* (T-t)} c(t) dt.$$

Taking $T = T_0$, we finally obtain the result. \square

Remark 5.1. In the preceding proof, the main changes comparing to [7] are essentially in the bound of J_x . Furthermore, we also gain some regularity hypotheses by avoiding the use of the derivative in time.

6 Technical tools

We give below a lemma that was used in the previous proof. Let us recall from [7] the following useful technical results:

Lemma 6.1. *Fix a function $\mu_1 \in \mathcal{C}_c^\infty(\mathbb{R}_+; \mathbb{R}_+)$ with*

$$\text{Supp}(\mu_1) \subseteq [0, 1[, \quad \int_{\mathbb{R}_+^*} r^{N-1} \mu_1(r) dr = \frac{1}{N\omega_N}, \quad \mu_1' \leq 0, \quad \mu_1^{(n)}(0) = 0 \text{ for } n \geq 1. \quad (6.1)$$

Define

$$\mu(x) = \frac{1}{\lambda^N} \mu_1\left(\frac{\|x\|}{\lambda}\right). \quad (6.2)$$

Then, recalling that $\omega_0 = 1$,

$$\int_{\mathbb{R}^N} \mu(x) dx = 1, \quad (6.3)$$

$$\int_{\mathbb{R}^N} |x_1| \mu_1(\|x\|) dx = \frac{2}{N} \frac{\omega_{N-1}}{\omega_N} \int_{\mathbb{R}^N} \|x\| \mu_1(\|x\|) dx, \quad (6.4)$$

$$\int_{\mathbb{R}^N} \|x\| \|\nabla \mu(x)\| dx = - \int_{\mathbb{R}^N} \|x\| \mu_1'(\|x\|) dx = N, \quad (6.5)$$

$$\int_{\mathbb{R}^N} \|x\|^2 \mu_1'(\|x\|) dx = -(N+1) \int_{\mathbb{R}^N} \|x\| \mu_1(\|x\|) dx. \quad (6.6)$$

Lemma 6.2. *Let I be defined as in (5.4). Then,*

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} I &\leq \int_{\|x-x_0\| \leq R+MT_0+\theta} |u(0, x) - v(0, x)| dx \\ &\quad - \int_{\|x-x_0\| \leq R+M(T_0-T)} |u(T, x) - v(T, x)| dx + 2 \sup_{\tau \in \{0, T\}} \text{TV}(u(\tau)) \lambda \\ &\quad + 2 \sup_{\substack{t \in \{0, T\} \\ s \in [t, t+\eta[}} \int_{\|y-x_0\| \leq R+\lambda+M(T_0-t)+\theta} |u(t, y) - u(s, y)| dy. \end{aligned}$$

Proof. See [7, Lemma 5.2]. \square

References

- [1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.
- [2] F. Bouchut and B. Perthame. Kružkov’s estimates for scalar conservation laws revisited. *Trans. Amer. Math. Soc.*, 350(7):2847–2870, 1998.
- [3] A. Bressan. *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000.
- [4] G.-Q. Chen and K. H. Karlsen. Quasilinear anisotropic degenerate parabolic equations with time-space dependent diffusion coefficients. *Commun. Pure Appl. Anal.*, 4(2):241–266, 2005.
- [5] R. M. Colombo, M. Herty, and M. Mercier. Control of the continuity equation with a non-local flow. *ESAIM Control Optim. Calc. Var.*, 2010.
- [6] R. M. Colombo and M. Lécureux-Mercier. A hyperbolic framework to describe the interaction between an individual and a continuum. *Preprint*, 2010.
- [7] R. M. Colombo, M. Mercier, and M. D. Rosini. Stability and total variation estimates on general scalar balance laws. *Communications in Mathematical Sciences*, 7(1):37–65, 2009.
- [8] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, second edition, 2005.
- [9] J. Dávila. On an open question about functions of bounded variation. *Calc. Var. Partial Differential Equations*, 15(4):519–527, 2002.
- [10] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.
- [11] B. J. Lucier. A moving mesh numerical method for hyperbolic conservation laws. *Math. Comp.*, 46(173):59–69, 1986.
- [12] D. Serre. *Systèmes de lois de conservation. I*. Fondations. Diderot Editeur, Paris, 1996.
- [13] D. Serre. *Systèmes de lois de conservation. II*. Fondations. Diderot Editeur, Paris, 1996.